ON PARASTATISTICS DEFINED AS TRIPLE OPERATOR ALGEBRAS

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Abstract

We unify parastatistics, defined as triple operator algebras represented on Fock space, in a simple way using the transition number operators. We express them as a normal ordered expansion of creation and annihilation operators. We discuss several examples of parastatistics, particularly Okubo's and Palev's parastatistics connected to many-body Wigner quantum systems. We relate them to the notion of extended Haldane statistics.

PACS No: 05.30.-d, 71.10.+x, 03.65 Bz

Abstract

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1. Introduction

Recently, a class of parastatistics (generalizing Bose and Fermi statistics) has been reformulated in terms of Lie supertriple systems [1]. Particularly, Green's parastatistics [2] as well as new kinds of parastatistics discovered by Palev [3,15] are reproduced. However, in this approach the positive definite Fock space representations are not treated.

On the other hand, a unified view of all operator algebras represented on Fock spaces has been presented [4]. Furthermore, the permutation invariant statistics are also studied in detail [5].

Along the lines of Refs.(4,5), in this paper we unify, in a simple way triple operator algebras of Ref.(1), represented on the Fock spaces, as well as Greenberg's infinite quon statistics [6] and Govorkov's paraquantization [7]. Particularly, we present and discuss parastatistics which naturally appears in many-body Wigner quantum systems [3] and its (bosonic and supersymmetric) extension[15]. It appears that they are a generalization of Klein-Marshalek algebra [8], extensively used in nuclear physics. We discuss them in the framework of the Haldane's definition of statistics [9]. We point out that none of them is an example of the original Haldane exclusion statistics, but can be related to the so-called extended Haldane statistics [10]. For each of them we find the extended Haldane statistics parameters .

2. Operator algebra, Fock space realization and statistics

Let us start with any algebra of M pairs of creation and annihilation operators $a_i^\dagger$, a_i, $i=1,2,..M$ ($a_i^\dagger$ is Hermitian conjugated to a_i). The algebra is defined by a normally ordered expansion Γ (generally no symmetry principle is assumed)

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger;a), \tag{1}$$

with the number operators N_i, i.e., $[N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}$, $[N_i, a_j] = -a_i\delta_{ij}$. In this case no peculiar relations of the type $a_i^m = a_j^n$, $i \neq j$ can appear. Then, every monomial in Γ_{ij}, Eq.(1), is of the type $(\cdots a_j^\dagger \cdots a_i \cdots)$ and all other indices appear in pairs $(\cdots a_k^\dagger \cdots a_k \cdots)$. The corresponding coefficients of expansion can depend on the total number operator $N=\sum_{i=1}^M N_i$.

We assume that there is a unique vacuum $|0>$ and the corresponding Fock space representation. The scalar product is uniquely defined by $<0|0>=1$, the vacuum condition $a_i|0>=0$, $i=1,2,..M$, and Eq.(1). A general N-particle state is a linear combination of the vectors $(a_{i_1}^\dagger \cdots a_{i_N}^\dagger|0>)$, $i_1,\cdots i_N = 1,2,...M$. We consider Fock spaces with no state vectors of negative squared norms. Note that we do not specify any relation between the creation (or annihilation) operators. They appear as a consequnce of the norm zero vectors (null-vectors) in Fock space.

For fixed N mutually different indices $i_1,\cdots i_N$, we define the $(N!\times N!)$ hermitian matrix of scalar products between states $(a_{i_{\pi(1)}}^\dagger \cdots a_{i_{\pi(N)}}^\dagger|0>)$ for all permutations $\pi \in S_N$. The number of linearly independent states among them is given by $d_{i_1,\cdots i_N} = rank\mathcal{A}(i_1,\cdots i_N)$. The set of $d_{i_1,\cdots i_N}$ for all possible $i_1,\cdots i_N$ and all integers N completely characterizes the statistics and the thermodynamic properties of a *free*

system with the corresponding Fock space [10].

If the algebra (1) is permutation invariant [5], i.e. $\langle \pi\mu|\pi\nu\rangle = \langle \mu|\nu\rangle$, for all $\pi, \mu, \nu \in S_N$, all expansion terms in Γ_{ij} of the form (symbolically)

$$\Gamma_{ij} := \sum (\underbrace{a^\dagger \cdots a^\dagger}_{l-r})(a_j^\dagger \underbrace{a^\dagger \cdots a^\dagger}_{r} \underbrace{a \cdots a}_{s} a_i)(\underbrace{a \cdots a}_{l-s})$$

have the same coefficient for all $i, j = 1, 2, ...M$. (One single relation, for example $a_1 a_2^\dagger = \Gamma_{12}$, determines the whole algebra.)

For the permutation invariant algebras there are several important consequences [5].

Consequences

(i) The matrices $\mathcal{A}(i_1, \cdots i_N)$ and their ranks do not depend on concrete indices $i_1, \cdots i_N$, but only on the multiplicities λ_i of appearance of the same indices $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_M \geq 0$, $|\lambda| = \sum_{i=1}^M \lambda_i = N$, i.e. on the partition λ of N.

(ii) For mutually different indices $i_1, \cdots i_N$ i.e. $\lambda_1 = \lambda_2 = \cdots \lambda_N = 1$, the generic matrix $\mathcal{A}_{1^N}$ is

$$\mathcal{A}_{1^N} = \sum_{\pi \in S_N} c(\pi) R(\pi), \tag{2}$$

where R is the right regular representation of the permutation group S_N and $c(\pi)$ are (real) coefficients. In other words, any row (column) of the matrix determines the whole matrix $\mathcal{A}_{1^N}$.

(iii) All matrices $\mathcal{A}_\lambda$ can be simply obtained from $\mathcal{A}_{1^N}$ [5]. To check that the Fock space does not contain states of negative norms, it is sufficient to show that only generic matrices are non-negative [11].

(iv) For permutation invariant algebras there exist the transition number operators N_{ij} , $i, j = 1, 2...M$ with the properties

$$[N_{ij}, a_k^\dagger] = \delta_{ik}\, a_j^\dagger, \quad [N_{ij}, a_k] = -\delta_{jk}\, a_i, \quad N_{ij}^\dagger = N_{ji}, \quad N_{ii} \equiv N_i. \tag{3}$$

N_{ij} can be presented similarly as Γ_{ij}, i.e. as a normal ordered expansion

$$N_{ij} = a_j^\dagger a_i + \alpha \sum_l a_l^\dagger a_j^\dagger a_i a_l + \beta \sum_l (a_l^\dagger a_j^\dagger a_l a_i + a_j^\dagger a_l^\dagger a_i a_l) + \gamma \sum_l a_j^\dagger a_l^\dagger a_l a_i + \cdots, \tag{4}$$

where α, β, γ are constants which do not depend on the indices i, j.

In the next section we show that all permutation invariant statistics considered by Okubo [1], Palev [3,15], Greenberg [6], Govorkov [7] and Klein and Marshalek [8] can be simply unified in terms of triple-operator algebras

$$[[a_i, a_j^\dagger]_q, a_k^\dagger] = x\, \delta_{ij}\, a_k^\dagger + y\, \delta_{ik}\, a_j^\dagger + z\, \delta_{jk}\, a_i^\dagger, \tag{5}$$

for all $i, j, k = 1, 2, \cdots M$. Here, $x, y, z \in \mathbf{R}$ are constants, $[\ ,\ ]$ denotes the commutator and $[a, b]_q = ab - qba$ is the q-deformed commutator.

Remarks

1. Equation (1), together with the vacuum condition $a_i|0>= 0$, uniquely determines all matrices $\mathcal{A}_{1^N}$ and $\mathcal{A}_\lambda$. However, equation (1) does not imply positive definiteness, which has to be checked separately.

2. All other triple-operator relations follow from Eq.(5) via hermiticity of creation and annihilation operators, a linear combination of Eq.(5) with indices interchanged and, finally,as null-states of matrices $\mathcal{A}_\lambda$.

3. The algebra with the well defined number operators N_i imply that $z = 0$ in (5). If $z \neq 0$, there exist peculiar relations of the type $a_i^2 = a_j^2$ for all $i, j = 1, 2, \cdots M$,

although $a_i^\dagger|0\rangle$ are linearly independent states. Such peculiar algebras are consistent if the Fock space does not contain null-states [12].

4. We point out that the algebra (5) can be simply written as the normal ordered expansion

$$a_i a_j^\dagger = (1 + xN)\delta_{ij} + q a_j^\dagger a_i + y N_{ij} + z N_{ji}, \tag{6}$$

where N_{ij} are the transition number operator of form (3,4) and N is the total number operator.

3. Examples

Example 1. Green's parastatistics [2] can be presented in the form of Eq.(6) with $x = z = 0, y = \frac{2}{p}, p \in \mathbf{N}$ and $q = \pm 1$, i.e.

$$a_i a_j^\dagger = \delta_{ij} \mp a_j^\dagger a_i \pm \frac{2}{p} N_{ij}, \tag{7}$$

where the upper (lower) sign corresponds to para-Bose (para-Fermi) statistics. The transition number operator N_{ij} is, up to the second order, given by [4]

$$N_{ij} = a_j^\dagger a_i + \frac{p^2}{4(p-1)} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots, \tag{8}$$

where $Y_{il} = a_i a_l - q\,(\frac{2}{p} - 1)\, a_l a_i$.

Example 2. Govorkov's new paraquantization [7] is given by $x = z = 0$, $y = \frac{\lambda}{p}, \lambda = \pm 1, p \in \mathbf{N}$ and $q = 0$:

$$a_i a_j^\dagger = \delta_{ij} - \frac{\lambda}{p} N_{ij}, \tag{9}$$

with the transition number operator, up to the second order,

$$N_{ij} = a_j^\dagger a_i + \frac{p^2}{p^2-\lambda^2} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots, \tag{10}$$

where $Y_{il} = a_i a_l + (\frac{\lambda}{p})\, a_l a_i$.

Example 3. Greenberg's infinite quon statistics [6] is given by $x = y = z = 0$ and $-1 < q < 1$:

$$a_i a_j^\dagger = \delta_{ij} + q\, a_j^\dagger a_i \tag{11}$$

with transition number operator, up to the second order,

$$N_{ij} = a_j^\dagger a_i + \frac{1}{1-q^2} \sum_l [Y_{jl}]^\dagger [Y_{il}] + \cdots, \tag{12}$$

where $Y_{il} = a_i a_l - q\, a_l a_i$. (The closed form for N_{ij} to all orders and for the general parameter q_{ij} is presented in Ref.(13).)

Example 4. Palev's A statistics (Fermi case), which appears naturally in the treatment of many-body Wigner quantum systems [3], is described by the following algebra $(i, j, k = 1, 2, \cdots M)$:

$$\begin{aligned}
[\{a_i, a_j^\dagger\}, a_k^\dagger] &= \delta_{ik}\, a_j^\dagger - \delta_{ij}\, a_k^\dagger, \\
[\{a_i, a_j^\dagger\}, a_k] &= -\delta_{jk}\, a_i + \delta_{ij}\, a_k, \\
\{a_i, a_j\} &= \{a_i^\dagger, a_j^\dagger\} = 0.
\end{aligned} \tag{13}$$

Hereafter, $\{,\}$ denotes the anticommutator.

(In the original algebra, the operators depend on two indices, $a_i \mapsto a_{\alpha i}$, but the structure of the algebra depends on the single index. One recovers the original algebra with $\delta_{\alpha i, \beta j} = \delta_{\alpha\beta}\delta_{ij}$). The vacuum conditions are $a_i|0\rangle = 0$, $a_i a_j^\dagger|0\rangle = p\, \delta_{ij}\, |0\rangle$

for $p \in \mathbf{N}$. Upon the redefinition of the operators $(a_i, a_i^\dagger) \mapsto (\sqrt{p}a_i, \sqrt{p}a_i^\dagger)$, we write the above algebra as normal ordered expansion with $x = -\frac{1}{p}$, $y = \frac{1}{p}$, $z = 0$ and $q = -1$

$$a_i a_j^\dagger = (1 - \frac{N}{p})\delta_{ij} - a_j^\dagger a_i + \frac{1}{p} N_{ij}. \tag{14}$$

The action of the annihilation operators a_i on the Fock states is obtained from the above relation, Eq.(14). For example,

$$a_i a_j^\dagger a_k^\dagger |0\rangle = (1 - \frac{1}{p})(\delta_{ij}\, a_k^\dagger - \delta_{ik}\, a_j^\dagger)|0\rangle.$$

It follows that

$$\begin{aligned} a_i (a_i^\dagger)^2 |0\rangle &= 0, \qquad \forall i \\ a_i a_i^\dagger a_k^\dagger |0\rangle = -a_i a_k^\dagger a_i^\dagger |0\rangle &= (1 - \tfrac{1}{p})\, a_k^\dagger |0\rangle, \quad i \neq k. \end{aligned} \tag{15}$$

Hence, we obtain $\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$.

Generally, for mutually different indices $i_1, \cdots i_N$, we find

$$a_{i_1} a_{i_1}^\dagger a_{i_2}^\dagger \cdots a_{i_N}^\dagger |0\rangle = (1 - \frac{N-1}{p})\, a_{i_2}^\dagger \cdots a_{i_N}^\dagger |0\rangle, \tag{16}$$

in accordance with Ref.(3). The Fock space does not contain negative norm states if $p \in \mathbf{N}$. The above equation (16) implies that the allowed states are only those with $N \leq p$, and the states with $N > p$ are null-states.

The transition number operartor N_{ij}, up to the second order, is:

$$N_{ij} = a_j^\dagger a_i + \frac{1}{(p-1)} \sum_l a_l^\dagger a_j^\dagger a_i a_l + \frac{2}{(p-1)(p-2)} \sum_{l_1, l_2} a_{l_2}^\dagger a_{l_1}^\dagger a_j^\dagger a_i a_{l_1} a_{l_2} + \cdots \tag{17}$$

and terminates with p creation and p annihilation operator terms. For example, if $p = 2$, the terms with $(p-2)$ appearing in the denominator do not appear at all.

The $p \to \infty$ reproduces the Fermi algebra. We note that case $p = 1$ reproduces the Klein-Marshalek algebra [8], namely only the one-particle states are allowed:

$$a_i a_j^\dagger = (1 - N)\, \delta_{ij}, \qquad N = \sum_l a_l^\dagger a_l. \tag{18}$$

In this sense, algebra (14) generalizes the Klein-Marshalek algebra.

It is interesting that the Fock space generated by the algebra (14) is equivalent to the Fock space generated by the algebra (with the same vacuum condition imposed)

$$a_i a_j^\dagger = (1 - \frac{N}{p})\, (\delta_{ij} - a_j^\dagger a_i), \tag{19}$$

with the same N_{ij} and N as given by Eq.(17).

Furthermore, there are infinitely many algebras leading to different generic matrices, but with the same statistics. They can be represented by

$$a_i a_j^\dagger = f(N)(\delta_{ij} - a_j^\dagger a_i), \tag{20}$$

with $f(n) > 0$, $n < p$ and $f(p) = 0$. The simplest choice is the step function $f(N) = \Theta(p - N)$ ($\Theta(x) = 0$, $x \le 0$ and $\Theta(x) = 1$, $x > 0$).

We point out that the corresponding statistics is Fermi statistics restricted up to $N \le p$ N-particle states. Hence, the counting rule is simply $D^F(M, N) = \left(\begin{array}{c} M \\ N \end{array} \right)$, $N \le p$ and $D^F(M, N) = 0$ if $N > p$. Recall that Haldane [9] introduced the statistics parameter g through the change of the single-particle *Hilbert* space dimension d_n

$$g_{n \to n+\Delta n} = \frac{d_n - d_{n+\Delta n}}{\Delta n},$$

where n is the number of particles and d_n is the dimension of the one-particle Hilbert space obtained by keeping the quantum numbers of $(n - 1)$ particles fixed. In the

similar way we define the *extended* statistics parameter through the change of the available one-particle *Fock*-subspace dimension [10]. Therefore, the above statistics is characterized by the Haldane statistical parameter $g = 1$

$$g_{n \to n+k} = \frac{d_n - d_{n+k}}{k} = \frac{(M-n+1)-(M-n-k+1)}{k} = 1, \tag{21}$$

if $n+k \le p$. If $n+k = p+1$, then $g_{n \to n+k} = \frac{(M-n+1)}{(p-n+1)}$, $n = 1, 2, \cdots p$ is fractional but g is not constant any more. Hence, this is not an example for the original Haldane statistics for which the statistics parameter is $g = const$. Moreover, the above statistics is also not the statistics of the Karabali-Nair type [14], where $a_i^p \neq 0$, $a_i^{p+1} = 0$, and for any $N \le Mp$ N-particle state is allowed, since from the Eq.(15) we already have $a_i^2 = 0$ and $N \le p$.

Example 5. Palev's A statistics[15] (Bose case) is the counterpart of the algebra (14), namely:

$$\begin{gathered}
[[a_i, a_j^\dagger], a_k^\dagger] = -\delta_{ik}\, a_j^\dagger - \delta_{ij}\, a_k^\dagger, \\
[[a_i, a_j^\dagger], a_k] = \delta_{jk}\, a_i + \delta_{ij}\, a_k, \\
[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \qquad i, j, k = 1, 2, \cdots M.
\end{gathered} \tag{22}$$

and the vacuum condition $a_i a_j^\dagger |0\rangle = p\, \delta_{ij}\, |0\rangle$. After the redefinition of the operators $(a_i, a_i^\dagger) \mapsto (\sqrt{p} a_i, \sqrt{p} a_i^\dagger)$, we write the normal ordered expansion of $a_i a_j^\dagger$ as
($x = y = -\frac{1}{p}$, $z = 0$, $q = -1$)

$$a_i a_j^\dagger = (1 - \frac{N}{p})\delta_{ij} + a_j^\dagger a_i - \frac{1}{p}\, N_{ij}. \tag{23}$$

The action of the annihilation operators a_i on the Fock states is obtained from

Eq.(23). For example,

$$a_i a_j^\dagger a_k^\dagger |0\rangle = (1 - \frac{1}{p})(\delta_{ij}\, a_k^\dagger + \delta_{ik}\, a_j^\dagger)|0\rangle.$$

Hence, we obtain $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$.

Generally, we find

$$a_i (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_M^\dagger)^{n_M} |0\rangle = N_i\, (1 - \frac{N-1}{p})\, (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \cdots (a_i^\dagger)^{n_i - 1} \cdots a_M^\dagger)^{n_M} |0\rangle \tag{24}$$

where $N = \sum_{i=1}^M n_i$. The Fock space does not contain negative norm states if $p \in \mathbf{N}$. The above equation (24) implies that the states with $N \leq p$ are allowed and the states with $N > p$ are null-states. The transition number operator N_{ij} has the same form, with the same coefficients as in the Fermi case, Eq.(17), and terminates with p-annihilation and p-creation operator terms. The limit $p \to \infty$ reproduces the Bose algebra. We note that if $p = 1$ the above algebra (23) reproduces the Klein-Marshalek algebra [8]. Hence, this algebra is the Bose generalization of the Klein-Marshalek algebra.

There are again infinitely many algebras leading to different generic matrices but of the same ranks, i.e. statistics. They can be represented by

$$a_i a_j^\dagger = f(N)(\delta_{ij} + a_j^\dagger a_i), \tag{25}$$

with $f(n) > 0$, $n < p$ and $f(p) = 0$. The simplest choice is the step function mentioned after Eq.(20) or $f(N) = 1 - \frac{N}{p}$. The corresponding statistics is Bose statistics restricted to N-particle states with $N \leq p$. Hence, the counting rule is simply $D^B(M,N) = \begin{pmatrix} M+N-1 \\ N \end{pmatrix}$, $N \leq p$ and $D^B(M,N) = 0$ if $N > p$. Therefore, the above statistics is characterized by the Haldane statistics parameter $g = 0$

$$g_{n\to n+k} = \frac{d_n - d_{n+k}}{k} = \frac{M-M}{k} = 0, \tag{26}$$

if $n+k \le p$. If $n+k = p+1$, then $g_{n\to n+k} = \frac{M}{(p-n+1)}$, $n = 1,2,\cdots p$, is fractional but not constant . Hence, this is not an example for the original Haldane exclusion statistics for which g should be constant. The above statistics is also not of the Karabali-Nair type [14], since $a_i^p \neq 0$, $a_i^{p+1} = 0$ but $N \le p$. This would be equivalent only for the single-mode oscillator, $M = 1$.

Example 6. The Bose and Fermi restricted algebra of Refs. (1,3) (the super-triple system) can be defined as

$$[a_I, a_J^\dagger]_q = (1 - \frac{N}{p})\delta_{IJ} - \frac{(-)^{\sigma(I)\sigma(J)}}{p} N_{IJ}, \tag{27}$$

$$q = (-)^{\sigma(I)\sigma(J)},$$

$$\sigma(I) = \begin{cases} 0 & \text{if } I = i \quad \text{(Bose)} \\ 1 & \text{if } I = \alpha \quad \text{(Fermi)} \end{cases}$$

where the index $I \doteq (i = 1,2,\cdots M_B; \alpha = 1,2,\cdots M_F.)$ denotes bosonic (fermionic) oscillator and $N = N_B + N_F$ is the total number operator.

Explicitly,

$$[a_i, a_j^\dagger] = (1 - \frac{N}{p})\delta_{ij} - \frac{1}{p}N_{ij},$$

$$\{a_\alpha, a_\beta^\dagger\} = (1 - \frac{N}{p})\delta_{\alpha\beta} + \frac{1}{p}N_{\alpha\beta}$$

$$[a_i, a_\alpha^\dagger] = -\frac{1}{p}N_{i\alpha}$$

$$[a_\alpha, a_i^\dagger] = -\frac{1}{p}N_{\alpha i}.$$

The consistency condition for the algebra (27) reads:

$$N_{IJ}a_K^\dagger - (-)^{(\sigma(I)+\sigma(J))\sigma(K)}a_K^\dagger N_{IJ} = \delta_{IK}a_J^\dagger. \qquad (28)$$

For example,

$$\{N_{i\alpha}, a_\gamma\} = +\delta_{\alpha\gamma}a_i,$$

$$[N_{i\alpha}, a_j^\dagger] = \delta_{ij}a_\alpha^\dagger.$$

Notice that $(N_{i\alpha})^2 = 0$. Thus, $N_{i\alpha}$ plays the role of supersymmetric charge. Furthermore, it follows that

$$[a_i, a_j] = \{a_\alpha, a_\beta\} = [a_i, a_\alpha] = 0.$$

The action of the annihilation operators a_i, a_α on the Fock states is obtained by combining Eqs.(27) and (28). The N-particle states are allowed only if $N \leq p$, with p being an integer.

The transition number operators, up to the second order, are basically similar to (17) and read

$$N_{IJ} = a_J^\dagger a_I + \frac{1}{(p-1)}\sum_L (-)^{\sigma(L)(\sigma(I)+\sigma(J))}a_L^\dagger a_J^\dagger a_I a_L + \qquad (29)$$
$$+\frac{2}{(p-1)(p-2)}\sum_{L_1,L_2}(-)^{(\sigma(L_1)+\sigma(L_2))(\sigma(I)+\sigma(J))}a_{L_2}^\dagger a_{L_1}^\dagger a_J^\dagger a_I a_{L_1} a_{L_2} + \cdots,$$

where the sum over L runs over bosonic $(i = 1, 2, \cdots M_B)$ and fermionic $(\alpha = 1, 2, \cdots M_F)$ indices.

In the limit $p \to \infty$, the above algebra reduces to the ordinary Bose and Fermi algebra. If $p = 1$, the above algebra reduces to the Klein - Marshalek algebra with $M_B + M_F$ oscillators.

Example 7. Okubo's triple operator algebra (Example 4. in Ref.(1)) is defined for the fermionic operators a_i as

$$[\{a_i, a_j^\dagger\}, a_k^\dagger] = (\frac{2}{p})(-\delta_{ij}a_k^\dagger - \delta_{jk}a_i^\dagger + \delta_{ik}a_j^\dagger). \tag{30}$$

The normal ordered expansion of $a_i a_j^\dagger$ is given by ($x = z = -\frac{2}{p}$, $y = \frac{2}{p}$, $q = -1$)

$$a_i a_j^\dagger = (1 - \frac{2N}{p})\delta_{ij} - a_j^\dagger a_i + (\frac{2}{p})(N_{ij} - N_{ji}). \tag{31}$$

In the limit $p \to \infty$, it becomes the Fermi algebra.

From (31) it follows that

$$\begin{aligned} a_i(a_j^\dagger)^2|0\rangle &= -(\tfrac{2}{p})a_i^\dagger|0\rangle, \qquad \forall i, j \\ a_i a_i^\dagger a_k^\dagger|0\rangle = -a_i a_k^\dagger a_i^\dagger|0\rangle &= (1 - \tfrac{2}{p})\, a_k^\dagger|0\rangle \quad i \neq k. \end{aligned} \tag{32}$$

Therefore,

$$\begin{gathered} \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0, \qquad i \neq j, \\ (a_i)^2 = A, \qquad [a_i, A] = 0, \quad [a_i, A^\dagger] = -(\tfrac{2}{p})a_i^\dagger, \quad \forall i, \\ (a_i)^p \neq 0, \qquad (a_i)^{p+1} = 0. \end{gathered} \tag{33}$$

However, in the Fock space there are negative norm states since $\langle 0|(a_i)^2(a_i^\dagger)^2|0\rangle = -(\frac{2}{p}) < 0$. The necessary condition for absence of such states is $z \geq 0$. The algebra similar to the algebra described by Eqs.(31-33) but with the positive definite Fock representations has been called peculiar algebra and was studied in Ref.(12).

Finally, let us mention that all Lie (super) algebras are triple systems (since $[a_i, a_j^\dagger]_\pm = \delta_{ij}(c_i + d_i N_i)$) and for a irreducible representations characterized with highest (lowest) weight state Λ ("vacuum") one can find the following normal ordered expansion

$$a_i a_i^\dagger = \Gamma_i(a^\dagger, a; \Lambda), \qquad a_i a_j^\dagger = \pm a_j^\dagger a_i$$

However, these systems are not permutation invariant in the sense we defined in this paper.

Acknowledgement

We thank T.Palev for providing us with references (3,15) and interesting remarks.

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